DIRECTED IMMERSIONS OF CLOSED MANIFOLDS

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Abstract. Given any finite subset X of the sphere $\mathbf{S}^n$, $n \geq 2$, which includes no pairs of antipodal points, we explicitly construct smoothly immersed closed orientable hypersurfaces in Euclidean space $\mathbf{R}^{n+1}$ whose Gauss map misses X. In particular, this answers a question of M. Gromov.

1. Introduction

To every ($\mathcal{C}^1$) immersion $f\colon M^n \to \mathbf{R}^{n+1}$ of a closed oriented n-manifold M, there corresponds a unit normal vector field or *Gauss map* $G_f\colon M \to \mathbf{S}^n$, which generates a set $G_f(M) \subset \mathbf{S}^n$ known as the *spherical image* of f. Conversely, one may ask, c.f. [8, p. 3]: *for which sets $A \subset \mathbf{S}^n$ is there an immersion $f\colon M \to \mathbf{R}^{n+1}$ such that $G_f(M) \subset A$?* Such a mapping would be called an *A-directed immersion* of M [1, 7, 13, 14]. It is well-known that when $A \neq \mathbf{S}^n$, f must have double points (Note 4.1), and M must be parallelizable, e.g., M can only be the torus $\mathbf{T}^2$ when $n = 2$ (Note 4.2). Furthermore, the only known necessary condition on A is the elementary observation that $A \cup -A = \mathbf{S}^2$, while there is also a sufficient condition due to Gromov [7, Thm. (D'), p. 186]:

Condition 1.1. *$A \subset \mathbf{S}^n$ is open, and there is a point $p \in \mathbf{S}^n$ such that the intersection of A with each great circle passing through p includes a (closed) semicircle.*

A *great circle* is the intersection of $\mathbf{S}^n$ with a 2-dimensional subspace of $\mathbf{R}^{n+1}$. Note that, when $n \geq 2$, examples of sets $A \subset \mathbf{S}^n$ satisfying the above condition include those which are the complement of a finite set of points without antipodal pairs. Thus the spherical image of a closed hypersurface can be remarkably flexible. Like most h-principle or convex integration type arguments, however, the proof does not yield specific examples. It is therefore natural to ask, for instance:

Question 1.2 ([7], p. 186)**.** *"Is there a 'simple' immersion $\mathbf{T}^2 \to \mathbf{R}^3$ whose spherical image misses the four vertices of a regular tetrahedron in $\mathbf{S}^2$?"*

Here we give an affirmative answer to this question (Section 2), and more generally present a short constructive proof of the sufficiency of a slightly stronger version of Condition 1.1 for the existence of A-directed immersions of parallelizable manifolds

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$M^{n-1}\times\mathbf{S}^1$, where M^{n-1} is closed and orientable. Any such manifold admits an immersion $f\colon M^{n-1}\to\mathbf{R}^n\times\{0\}\subset\mathbf{R}^{n+1}$ (Note 4.3). We then extend f to $M^{n-1}\times\mathbf{S}^1$ by using the *figure eight curve*

$$E_\delta(t) := \big(\cos(t),\ \delta\sin(2t)\big) \tag{1}$$

to put a copy of $\mathbf{S}^1\simeq\mathbf{R}/2\pi$ in each normal plane of f, as described below. Note that the midpoint of $G_{E_\delta}(\mathbf{S}^1)$ is assumed to be at $(1,0)$; see Figure 1 which shows $E_{1/2}$ and its spherical image. Further, the unit normal bundle of f may be naturally identified with the pencil of great circles of $\mathbf{S}^n$ passing through $(0,\dots,0,1)$.

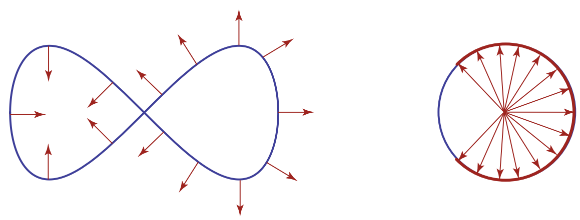

FIGURE 1.

Theorem 1.3. *Let $A\subset\mathbf{S}^n$ satisfy Condition 1.1 with respect to $p=(0,\dots,0,1)$. Further, if $n\geq 3$, suppose that the semicircle in Condition 1.1 contains p, or that no great circle through p is contained in A. Let $f\colon M^{n-1}\to\mathbf{R}^n\times\{0\}\subset\mathbf{R}^{n+1}$ be a smooth ($\mathcal{C}^\infty$) immersion of a closed orientable $(n-1)$-manifold, and, for every $q\in M$, $C_q\subset\mathbf{S}^n$ be the unit normal space of f at q. Then there is a smooth orthogonal frame $\{N_i\colon M\to\mathbf{S}^n\}$, $i=1,2$, for the normal bundle of f such that the semicircle in C_q centered at $N_1(q)$ lies in A. For any such frame, and sufficiently small ε, $\delta>0$,*

$$F(q,t) := f(q)+\varepsilon\sum_{i=1}^{2}E^i_\delta(t)N_i(q) \tag{2}$$

yields a smooth A-directed immersion $M\times\mathbf{S}^1\to\mathbf{R}^{n+1}$, where E^i_δ are the components of the figure eight curve E_δ given by (1).

It is not known if Condition 1.1 is necessary for the existence of A-directed closed hypersurfaces, and the question posed in the first paragraph is open, even for $n=2$. See [3, 4, 6] for some other recent results on Gauss maps of closed submanifolds, [2, 9, 11, 16] for still more studies of spherical images, and [15] for historical background.

2. EXAMPLE

If $A=\mathbf{S}^2\smallsetminus X$ for a finite set X without antipodal pairs, we may always find a point $p\in\mathbf{S}^2$ with respect to which A satisfies the hypothesis of Theorem 1.3 (e.g., let $p\notin X$ be in the complement of all great circles which pass through at least two points of X other than $-p$). After a rigid motion (which may be arbitrarily small)

we may assume that $p=(0,0,1)$ or $(0,0,-1)$, and let $f(\theta):=(\cos(\theta),\sin(\theta),0)$ be the standard immersion of $\mathbf{S}^1\simeq\mathbf{R}/2\pi$ in $\mathbf{R}^3$. Then the desired framing for the normal bundle of f may always take the form

$$N_1(\theta):=f'(\theta)\times N_2(\theta),\qquad N_2(\theta):=\frac{\big(\cos(\theta),\,\sin(\theta),\,z(\theta)\big)}{\sqrt{1+z^2(\theta)}},\tag{3}$$

where $z\colon\mathbf{R}/2\pi\to\mathbf{R}$ is a smooth function with $z(\theta)=-z(\theta+\pi)$ and such that X is contained entirely in one of the components of $\mathbf{S}^2-N_2(\mathbf{S}^1)$. For instance, when X is the vertices of a regular tetrahedron, we may set $z(\theta):=\cos(3\theta)$ in (3). Then, for ε, $\delta\le 1/8$, the mapping $F(\theta,t)$ given by (2) yields an immersion $\mathbf{T}^2\simeq\mathbf{R}/2\pi\times\mathbf{R}/2\pi\to\mathbf{R}^3$ which answers Question 1.2. The resulting surface, for $\varepsilon=\delta=1/8$, is depicted in Figure 2 together with its spherical image (note that

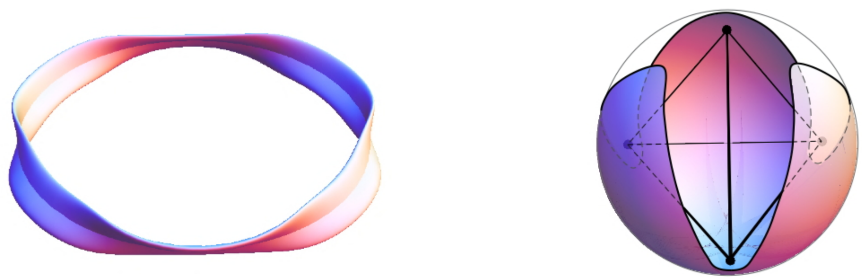

FIGURE 2.

here $p=(0,0,-1)$). To find $z(\theta)$ in general, we may order the points in $X'\cup -X'$, where $X':=X\smallsetminus\{-p\}$, according to their "longitude" θ, and connect them by geodesic segments to obtain a simple closed symmetric curve $\gamma(\theta)$. A perturbation of γ then yields a smooth symmetric curve $\widetilde{\gamma}$ such that X is contained in one of the components of $\mathbf{S}^2-\widetilde{\gamma}(\mathbf{S}^1)$. The third coordinate of $\widetilde{\gamma}$ gives our desired height function z.

3. Proof of Theorem 1.3

3.1. First we construct the frame $\{N_i\}$. For every $q\in M$, C_q is a great circle passing through p. So it contains a semicircle in A by assumption (Condition 1.1). Let $m_q\subset C_q$ be the set of midpoints of all such semicircles. We need to find a smooth map $N_1\colon M\to\mathbf{S}^n$ such that $N_1(q)\in m_q$ for all $q\in M$. To this end note that m_q is open and connected. Further, if m_q contains any pairs of antipodal points, then $m_q=C_q$; otherwise, m_q lies in the interior a semicircle of C_q. Consequently,

$$\operatorname{Cone}(m_q):=\{\,\lambda x\mid x\in m_q,\ \text{and}\ \ \lambda\ge 0\,\},$$

is a convex set in $\mathbf{R}^{n+1}$. In particular, for any finite set of points $x_i\in\operatorname{Cone}(m_q)$ and numbers $\lambda_i\ge 0$, $\sum_i\lambda_i x_i\in\operatorname{Cone}(m_q)$. Now let B be the set of all points $q\in M$ such that $m_q\neq C_q$. Then B is closed (and therefore compact) since $M\smallsetminus B$ is open; indeed the set of great circles contained in A is open, since A is open. Further note

that for any point $q \in M$, normal vector $x \in m_q$, and continuous local extension v of x to a normal vector field of M, we have $v(q') \in m_{q'}$ for all q' in an open neighborhood U of q (because the set of semicircles contained in A is open). Let $\{v_i \colon U_i \to \mathbf{S}^n\}$, $i = 1, \dots, k$, be a finite collection of such local vector fields so that $\cup_i U_i$ covers B and v_i are smooth. Also let $\{\phi_i \colon M \to \mathbf{R}\}$ be a smooth partition of unity subordinate to $\{U_i\}$, and, for $q \in \cup_i U_i$, set

$$N_1(q) := \frac{\sum_{i=1}^k \phi_i(q) v_i(q)}{\| \sum_{i=1}^k \phi_i(q) v_i(q)\|}.$$

If $q \in B$, then $v_i(q) \in m_q$ which lies in the interior of a semicircle $S \subset C_q$, and so $\| \sum_{i=1}^k \phi_i(q) v_i(q)\| \neq 0$. Indeed, if x is the midpoint of S, then $\langle \sum_{i=1}^k \phi_i(q) v_i(q), x\rangle = \sum_{i=1}^k \phi_i(q) \big\langle v_i(q), x\big\rangle > 0$. Thus N_1 is well defined (and smooth) on an open neighborhood V of B. Further, $N_1(q) \in m_q$, for all $q \in V$, since $\mathrm{Cone}(m_q)$ is convex. In particular we are done if $B = M$; otherwise, note that we may write

$$N_1(q) = \cos\big(\theta(q)\big)\, p + \sin\big(\theta(q)\big)\, G_f(q), \tag{4}$$

for some function $\theta \colon V \to \mathbf{R}$, since G_f is well defined due to the orientability of M, and thus $\{p, G_f(q)\}$ forms an orthonormal basis for the normal plane $df(T_qM)^\perp$. Further, it is easy to see that we may choose θ continuously (and therefore smoothly) if $n = 2$. This also holds for $n > 2$ if each C_q contains a semicircle passing through p; for then θ is uniquely determined within the range $[-\pi/2, \pi/2]$. Indeed, we may choose the vectors v_i above so that $\langle v_i(q), p\rangle \geq 0$ which would in turn yield that $\langle N_1(q), p\rangle \geq 0$. Now let V' be an open neighborhood of B with closure $\overline{V'} \subset V$. Using Tietze's theorem, followed by a perturbation and a gluing, we may extend $\theta|_{V'}$ smoothly to all of M. Then (4) yields the desired vector field on M, since for any $q \in M \setminus B$, $N_1(q) \in C_q = m_q$. Finally, set

$$N_2(q) := \sin\big(\theta(q)\big)\, p - \cos\big(\theta(q)\big)\, G_f(q).$$

3.2. It remains to show that $G_F(M \times \mathbf{S}^1) \subset A$, for small ε, $\delta > 0$. For all $q \in M$, $C_q \cap A$ contains an arc of length $\geq \pi + \alpha$ with midpoint $N_1(q)$ for some uniform constant $\alpha > 0$. Indeed, if we let $g(q)$ be the supremum of lengths of all arcs in $C_q \cap A$ with midpoint $N_1(q)$, then $g \colon M \to \mathbf{R}$ is lower semicontinuous, i.e., $\lim_{q \to q_0} g(q) \geq g(q_0)$, since A is open. Thus, since $g > \pi$ and M is compact, $g \geq \pi + \alpha$. Now choose $\delta > 0$ so small that the length ℓ of the spherical image of E_δ is $\leq \pi + \alpha$ (this is possible since $\ell \to \pi$ as $\delta \to 0$). Next, for $(q,t) \in M \times \mathbf{S}^1$, let $\widetilde{G}_F(q,t)$ be the normalized projection of $G_F(q,t)$ into $df(T_qM)^\perp$, i.e.,

$$\widetilde{G}_F(q,t) := \frac{\sum_{i=1}^2 \Big\langle G_F(q,t), N_i(q)\Big\rangle N_i(q)}{\sqrt{\sum_{i=1}^2 \Big\langle G_F(q,t), N_i(q)\Big\rangle^2}}.$$

Also, for fixed $t \in \mathbf{S}^1$, let $F_t(q) := F(q,t)$. Then, by the tubular neighborhood theorem, $F_t \colon M \to \mathbf{R}^{n+1}$ is a smooth immersion for small ε. Further, as $\varepsilon \to 0$, F_t converges to f with respect to the $\mathcal{C}^1$-topology. Thus, for each $q \in M$, the normal

plane $dF_t(T_qM)^\perp$ (which contains $G_F(q,t)$) converges to $df(T_qM)^\perp$. Consequently G_F is well-defined for small ε, and converges to $\widetilde{G}_F$ as $\varepsilon \to 0$. So it suffices to check that $\widetilde{G}_F(M \times \mathbf{S}^1) \subset A$, which follows from our choice of δ. Indeed for each fixed $q \in M$, $\widetilde{G}_F(\{q\} \times \mathbf{S}^1)$ is the spherical image of the figure eight curve $\sum_{i=1}^2 E^i_\delta(t)N_i(q)$ in $df(T_qM)^\perp$, which is an arc of C_q with midpoint $N_1(q)$ and length $\leq \pi + \alpha$. $\square$

4. Notes

4.1. It is well-known that $G_f(M) = \mathbf{S}^n$ for any embedding $f\colon M^n \to \mathbf{R}^{n+1}$ of a closed oriented n-manifold [7, p. 187]. More generally, this also holds for "Alexandrov embeddings", i.e., immersions $f\colon M \to \mathbf{R}^{n+1}$ which may be extended to an immersion $\overline{f}\colon \overline{M} \to \mathbf{R}^{n+1}$ of a compact $(n+1)$-manifold $\overline{M}$ with $\partial\overline{M} = M$. Indeed if v is any vector field along M which points "outward" with respect to $\overline{M}$, then for $p \in M$, the normalized projection of $df(v(p))$ into the line $df(T_pM)^\perp$ defines a normal vector field $M \to \mathbf{S}^n$ which coincides with G_f (after a reflection of G_f if necessary). Then, for any $u \in \mathbf{S}^n$, if p is a point which maximizes the height function $\langle \cdot, u\rangle$ on M, we have $G_f(p) = u$. On the other hand, being only regularly homotopic to an embedding, is not enough to ensure that $G_f(M) = \mathbf{S}^n$. Indeed the example in Figure 2 is regularly homotopic to an embedded torus of revolution [12].

4.2. If $G_f(M) \neq \mathbf{S}^n$ for an immersion $f\colon M^n \to \mathbf{R}^{n+1}$ of an oriented n-manifold, then, as is well-known [11], M must be parallelizable. Here we include a brief geometric argument for this fact. If $(0,\dots,0,1) \notin G_f(M)$, we may define a continuous map $F\colon TM \to \mathbf{R}^n \simeq \mathbf{R}^n \times \{0\} \subset \mathbf{R}^{n+1}$ as follows, c.f. [5, Lemma 2.2]. There is a continuous map $\mathbf{S}^n \smallsetminus \{(0,\dots,0,1)\} \overset{\rho}{\to} SO(n+1)$, $u \mapsto \rho_u$ such that $\rho_u(u) = (0,\dots,0,-1)$. Let $\pi\colon TM \to M$ be the canonical projection, and for $X \in TM$ set $F(X) := \rho_{G_f(\pi(X))}(df(X))$. Also let $F_p := F|_{T_pM}$. Then $\{F_p^{-1}(e_i)\}$, where $\{e_i\}$ is a fixed basis of $\mathbf{R}^n$, gives a framing for TM as desired. So in particular, when M is closed and $n = 2$, we have $M = \mathbf{T}^2$. The last observation also follows from Gauss-Bonnet theorem via degree theory when f is $\mathcal{C}^2$; since if $G_f(M) \neq \mathbf{S}^2$, then

$$0 = \deg(G_f) = \frac{1}{4\pi}\int_M \det(dG_f) = \frac{1}{4\pi}\int_M K = \frac{1}{2}\chi(M),$$

where K is the Gaussian curvature and χ is the Euler characteristic.

4.3. To generate some concrete examples of the immersions $f\colon M^{n-1} \to \mathbf{R}^n \simeq \mathbf{R}^n \times \{0\}$ in Theorem 1.3, note that if $f_0\colon M_0^{n-k-1} \to \mathbf{R}^{n-k} \times \{0\}$ is any immersion such that $f_0(M_0)$ is disjoint from the subspace $L := \mathbf{R}^{n-k-1} \times \{(0,0)\}$, then spinning f_0 about L yields an immersion $f_1\colon M_0 \times \mathbf{S}^1 \to \mathbf{R}^{n-k+1}$ given by

$$f_1(q,t) := \left[\begin{array}{c|cc} \mathrm{I} & \multicolumn{2}{c}{0} \\ \hline 0 & \cos(t) & \sin(t) \\ & -\sin(t) & \cos(t) \end{array}\right] \begin{bmatrix} f_0^1(q) \\ \vdots \\ f_0^{n-k}(q) \\ 0 \end{bmatrix},$$

where f_0^i are the components of f_0. Thus, for instance, one may inductively construct immersions of $\mathbf{S}^{n-k-1}\times\mathbf{T}^k$ in $\mathbf{R}^n$, for $k=1,\dots,n-1$. More generally, if $M^{n-1}\times\mathbf{S}^1$ is parallelizable, then so is the open manifold $M^{n-1}\times(0,1)$, which may be immersed in $\mathbf{R}^n$ [10] by the h-principle [7], or more specifically, the "holonomic approximation theorem" of Eliashberg-Mishachev [1, 5].

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